Models of Typed Lambda Calculi

Riccardo Pucella

Cornell University

PLDG, September 30, 2004

Typed Lambda Calculi

- Often presented starting with the lambda calculus
  - Type system imposed to classify “good” terms
- I present it as a multi-sorted algebraic theory
  - constants and operations over a carrier set
  - Equations between elements

Types

- Assume a set Σ of ground types
  - $1 \in \Sigma$
- $A \times B$ is a type if A, B are types
- $A \rightarrow B$ is a type if A, B are types

Terms

A typed lambda calculus is a set of terms $\mathcal{T}_A$, one per type A, and constants and operations $*, \mathsf{pair}, \mathsf{fst}, \mathsf{snd}, \mathsf{app}, \lambda$ of the appropriate type

- Variables $x_1^A, x_2^A, \dots \in \mathcal{T}_A$
- $* \in \mathcal{T}_1$
- $\mathsf{pair}_{A,B}(a, b) \in \mathcal{T}_{A \times B}$ if $a \in \mathcal{T}_A, b \in \mathcal{T}_B$
- $\mathsf{fst}_{A,B}(c) \in \mathcal{T}_A$ if $c \in \mathcal{T}_{A \times B}$
- $\mathsf{snd}_{A,B}(c) \in \mathcal{T}_B$ if $c \in \mathcal{T}_{A \times B}$
- $\mathsf{app}_{A,B}(f, a) \in \mathcal{T}_B$ if $f \in \mathcal{T}_{A \to B}, a \in \mathcal{T}_A$
- $\lambda x^A.\varphi(x) \in \mathcal{T}_{A \to B}$ if $\varphi(x) \in \mathcal{T}_B$

Equational Theory

Properties of the constants and operations are defined by equations

$$a \underset{\Gamma}{=} b$$

- $a, b \in \mathcal{T}_A$
- Γ is a sequence of variables $x_1, \ldots, x_k$
- Free variables of a, b are in Γ

Equational theory defined by inference rules and axioms

Inference Rules

(Add) $$\frac{a \underset{\Gamma}{=} b}{a \underset{\Gamma'}{=} b} \quad (\Gamma \sqsubseteq \Gamma')$$

(Substitution) $$\frac{a \underset{\Gamma}{=} b}{\mathsf{app}(f, a) \underset{\Gamma}{=} \mathsf{app}(f, b)}$$

$$\frac{\varphi(x) \underset{\Gamma, x}{=} \varphi'(x)}{\lambda x^A.\varphi(x) \underset{\Gamma}{=} \lambda x^A.\varphi'(x)}$$

⋮

+ transitivity, reflexivity, symmetry

Axioms

$$a \underset{\Gamma}{=} * \qquad \text{(for all } a \in \mathcal{T}_1\text{)}$$

$$\mathsf{fst}(\mathsf{pair}(a, b)) \underset{\Gamma}{=} a \qquad \text{(for all } a \in \mathcal{T}_A,\ b \in \mathcal{T}_B\text{)}$$

$$\mathsf{snd}(\mathsf{pair}(a, b)) \underset{\Gamma}{=} b \qquad \text{(for all } a \in \mathcal{T}_A,\ b \in \mathcal{T}_B\text{)}$$

$$\mathsf{pair}(\mathsf{fst}(c), \mathsf{snd}(c)) \underset{\Gamma}{=} c \qquad \text{(for all } c \in \mathcal{T}_{A\times B}\text{)}$$

$$\mathsf{app}(\lambda x^A.\varphi(x), a) \underset{\Gamma}{=} \varphi(a) \qquad \text{(for all } a \in \mathcal{T}_A \text{ subst. for } x\text{)}$$

$$\lambda x^A.(\mathsf{app}(f, x)) \underset{\Gamma}{=} f \qquad \text{(for all } f \in \mathcal{T}_{A\to B},\ x \notin \Gamma\text{)}$$

$$\lambda x^A.\varphi(x) \underset{\Gamma}{=} \lambda x'^A.\varphi(x') \qquad (x' \text{ subst. for } x)$$

Set-Theoretic Models

Given a typed lambda calculus, we give it an interpretation using sets of values (called domains)

- A type A correspond to a domain of values $\mathcal{D}_A$
- A term $a \in \mathcal{T}_A$ corresponds (essentially) to a value $[\![a]\!] \in \mathcal{D}_A$
- Equations hold:
  - If $a \underset{\Gamma}{=} b$, then $[\![a]\!], [\![b]\!]$ are the same element of $\mathcal{D}_A$

Interpretation of Types

Assign to every type A a set $\mathcal{D}_A$ of values

- Choose $\mathcal{D}_A$ for every $A \in \Sigma$
  - $\mathcal{D}_1 = \{d_*\}$ for some d_*
- $\mathcal{D}_{A \times B} = \mathcal{D}_A \times \mathcal{D}_B$
- $\mathcal{D}_{A \to B} = (\mathcal{D}_B)^{\mathcal{D}_A}$

Interpretation of Terms

Assign to every term $a \in \mathcal{T}_A$ an element of $\mathcal{D}_A$

- Problem: need to account for *free variables* in a
- Free variables act as parameters

Assume that each constant $c \in \mathcal{T}_A$ is associated with an element $[\![c]\!]$ in $\mathcal{D}_A$

- $[\![*]\!] \in \mathcal{D}_1$, i.e., $[\![*]\!] = d_*$

Associate to every term $a \in \mathcal{T}_A$ in context Γ a *function* from the domain of the free variables to $\mathcal{D}_A$

- If $\Gamma = x_1^{A_1}, \ldots, x_k^{A_k}$, let $[\![\Gamma]\!] = \mathcal{D}_{A_1} \times \cdots \times \mathcal{D}_{A_k}$
- For $a \in \mathcal{T}_A$, $[\![a]\!]_\Gamma$ will be a function from $[\![\Gamma]\!]$ to $\mathcal{D}_A$, to be defined shortly

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

Interpretation of Terms

$$[\![c]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_{\Gamma}(d_1, \ldots, d_k) = d_i$$

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma(d_1, \ldots, d_k) = ([\![a]\!]_\Gamma(d_1, \ldots, d_k), [\![b]\!]_\Gamma(d_1, \ldots, d_k))$$

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma(d_1, \ldots, d_k) = ([\![a]\!]_\Gamma(d_1, \ldots, d_k), [\![b]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_1([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma(d_1, \ldots, d_k) = ([\![a]\!]_\Gamma(d_1, \ldots, d_k), [\![b]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_1([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_2([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma(d_1, \ldots, d_k) = ([\![a]\!]_\Gamma(d_1, \ldots, d_k), [\![b]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_1([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_2([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma(d_1, \ldots, d_k) = [\![f]\!]_\Gamma(d_1, \ldots, d_k)([\![a]\!]_\Gamma(d_1, \ldots, d_k))$$

Interpretation of Terms

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma(d_1, \ldots, d_k) = ([\![a]\!]_\Gamma(d_1, \ldots, d_k), [\![b]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_1([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma(d_1, \ldots, d_k) = \pi_2([\![c]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma(d_1, \ldots, d_k) = [\![f]\!]_\Gamma(d_1, \ldots, d_k)([\![a]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Some Properties of Sets (I)

For any sets $X_1, \ldots, X_k$, can form the set $X_1 \times \cdots \times X_k$ of tuples of elements from $X_1, \ldots, X_k$, with associated projection functions

$$\pi_i : X_1 \times \cdots \times X_k \to X_i$$
$$\pi_i(x_1, \ldots, x_k) = x_i$$

Some Properties of Sets (I)

For any sets $X_1, \ldots, X_k$, can form the set $X_1 \times \cdots \times X_k$ of tuples of elements from $X_1, \ldots, X_k$, with associated projection functions

$$\pi_i : X_1 \times \cdots \times X_k \to X_i$$
$$\pi_i(x_1, \ldots, x_k) = x_i$$

if $\{f_i : Z \to X_i\}$ is a family of functions, define

$$\langle f_1, \ldots, f_k \rangle : Z \to X_1 \times \cdots \times X_k$$
$$\langle f_1, \ldots, f_k \rangle(z) = (f_1(z), \ldots, f_k(z))$$

Some Properties of Sets (I)

For any sets $X_1, \ldots, X_k$, can form the set $X_1 \times \cdots \times X_k$ of tuples of elements from $X_1, \ldots, X_k$, with associated projection functions

$$\pi_i : X_1 \times \cdots \times X_k \to X_i$$
$$\pi_i(x_1, \ldots, x_k) = x_i$$

if $\{f_i : Z \to X_i\}$ is a family of functions, define

$$\langle f_1, \ldots, f_k \rangle : Z \to X_1 \times \cdots \times X_k$$
$$\langle f_1, \ldots, f_k \rangle(z) = (f_1(z), \ldots, f_k(z))$$

Theorem: $\langle f_1, \ldots, f_k \rangle$ is the *unique* function with

$$\pi_i(\langle f_1, \ldots, f_k \rangle(z)) = f_i(z) \qquad \text{(or } \pi_i \circ \langle f_1, \ldots, f_k \rangle = f_i\text{)}$$

We say *sets have finite products*

Element-Free Interpretation of Terms (I)

$$[\![c]\!]_\Gamma(d_1,\ldots,d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1,\ldots,d_k) = d_i$$

$$[\![\mathsf{pair}(a,b)]\!]_\Gamma(d_1,\ldots,d_k) = ([\![a]\!]_\Gamma(d_1,\ldots,d_k), [\![b]\!]_\Gamma(d_1,\ldots,d_k))$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma(d_1,\ldots,d_k) = \pi_1([\![c]\!]_\Gamma(d_1,\ldots,d_k))$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma(d_1,\ldots,d_k) = \pi_2([\![c]\!]_\Gamma(d_1,\ldots,d_k))$$

$$[\![\mathsf{app}(f,a)]\!]_\Gamma(d_1,\ldots,d_k) = [\![f]\!]_\Gamma(d_1,\ldots,d_k)([\![a]\!]_\Gamma(d_1,\ldots,d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1,\ldots,d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1,\ldots,d_k,d_{k+1}))$$

Element-Free Interpretation of Terms (I)

$$[\![c]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_{\Gamma}(d_1, \ldots, d_k) = d_i$$

$$[\![\mathsf{pair}(a, b)]\!]_{\Gamma} = \langle [\![a]\!]_{\Gamma}, [\![b]\!]_{\Gamma} \rangle$$

$$[\![\mathsf{fst}(c)]\!]_{\Gamma}(d_1, \ldots, d_k) = \pi_1([\![c]\!]_{\Gamma}(d_1, \ldots, d_k))$$

$$[\![\mathsf{snd}(c)]\!]_{\Gamma}(d_1, \ldots, d_k) = \pi_2([\![c]\!]_{\Gamma}(d_1, \ldots, d_k))$$

$$[\![\mathsf{app}(f, a)]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![f]\!]_{\Gamma}(d_1, \ldots, d_k)([\![a]\!]_{\Gamma}(d_1, \ldots, d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_{\Gamma}(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Element-Free Interpretation of Terms (I)

$$[\![c]\!]_\Gamma(d_1,\ldots,d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma(d_1,\ldots,d_k) = d_i$$

$$[\![\mathsf{pair}(a,b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f,a)]\!]_\Gamma(d_1,\ldots,d_k) = [\![f]\!]_\Gamma(d_1,\ldots,d_k)([\![a]\!]_\Gamma(d_1,\ldots,d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1,\ldots,d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1,\ldots,d_k,d_{k+1}))$$

Element-Free Interpretation of Terms (I)

$$[\![c]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_{\Gamma} = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_{\Gamma} = \langle [\![a]\!]_{\Gamma}, [\![b]\!]_{\Gamma} \rangle$$

$$[\![\mathsf{fst}(c)]\!]_{\Gamma} = \pi_1 \circ [\![c]\!]_{\Gamma}$$

$$[\![\mathsf{snd}(c)]\!]_{\Gamma} = \pi_2 \circ [\![c]\!]_{\Gamma}$$

$$[\![\mathsf{app}(f, a)]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![f]\!]_{\Gamma}(d_1, \ldots, d_k)([\![a]\!]_{\Gamma}(d_1, \ldots, d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_{\Gamma}(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Some Properties of Sets (II)

For any sets X and Y, can form the set Y^X of functions from X to Y, with associated evaluation functions

$$eval : Y^X \times X \to Y$$
$$eval(f, x) = f(x)$$

Some Properties of Sets (II)

For any sets X and Y, can form the set Y^X of functions from X to Y, with associated evaluation functions

$$eval : Y^X \times X \to Y$$
$$eval(f, x) = f(x)$$

If $g : Z \times X \to Y$ is a function, let $g_z : X \to Y$ be $g_z(x) = g(z, x)$. Define

$$curry(g) : Z \to Y^X$$
$$curry(g)(z) = g_z$$

Some Properties of Sets (II)

For any sets X and Y, can form the set Y^X of functions from X to Y, with associated evaluation functions

$$eval : Y^X \times X \to Y$$
$$eval(f, x) = f(x)$$

If $g : Z \times X \to Y$ is a function, let $g_z : X \to Y$ be $g_z(x) = g(z, x)$. Define

$$curry(g) : Z \to Y^X$$
$$curry(g)(z) = g_z$$

Theorem: $curry(g)$ is the *unique* function with

$$eval(curry(g)(z), x) = g(z, x) \qquad \text{(or } eval \circ (curry(g) \times id_{\ulcorner}) = g\text{)}$$

We say *sets have exponentials*

Element-Free Interpretation of Terms (II)

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma(d_1, \ldots, d_k) = [\![f]\!]_\Gamma(d_1, \ldots, d_k)([\![a]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Element-Free Interpretation of Terms (II)

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma(d_1, \ldots, d_k) = eval([\![f]\!]_\Gamma(d_1, \ldots, d_k), [\![a]\!]_\Gamma(d_1, \ldots, d_k))$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Element-Free Interpretation of Terms (II)

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma = eval \circ \langle [\![f]\!]_\Gamma, [\![a]\!]_\Gamma \rangle$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma(d_1, \ldots, d_k) = (d_{k+1} \mapsto [\![\varphi(x)]\!]_{\Gamma,x}(d_1, \ldots, d_k, d_{k+1}))$$

Element-Free Interpretation of Terms (II)

$$[\![c]\!]_\Gamma(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_\Gamma = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f, a)]\!]_\Gamma = eval \circ \langle [\![f]\!]_\Gamma, [\![a]\!]_\Gamma \rangle$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma = curry([\![\varphi(x)]\!]_{\Gamma,x})$$

Some Properties of Sets (III)

Fix a singleton set $T = \{x_0\}$ (any will do)

For any set X, define the function

$$t_X : X \to T$$
$$t_X(x) = x_0$$

Some Properties of Sets (III)

Fix a singleton set $T = \{x_0\}$ (any will do)

For any set X, define the function

$$t_X : X \to T$$
$$t_X(x) = x_0$$

Theorem: t_X is the *unique* function from X to T

We say *sets have a terminal object*

Some Properties of Sets (III)

Fix a singleton set $T = \{x_0\}$ (any will do)

For any set X, define the function

$$t_X : X \rightarrow T$$
$$t_X(x) = x_0$$

Theorem: t_X is the *unique* function from X to T

We say *sets have a terminal object*

Theorem: For any set Y, there is a one-one correspondence between elements of Y and the functions from T to Y

Instead of associating to every constant $c \in \mathcal{T}_A$ an element of $\mathcal{D}_A$, we associate a function $[\![c]\!]^{\rightarrow} : T \rightarrow \mathcal{D}_A$

Element-Free Interpretation of Terms (III)

$$[\![c]\!]_{\Gamma}(d_1, \ldots, d_k) = [\![c]\!]$$

$$[\![x_i]\!]_{\Gamma} = \pi_i$$

$$[\![\mathsf{pair}(a, b)]\!]_{\Gamma} = \langle [\![a]\!]_{\Gamma}, [\![b]\!]_{\Gamma} \rangle$$

$$[\![\mathsf{fst}(c)]\!]_{\Gamma} = \pi_1 \circ [\![c]\!]_{\Gamma}$$

$$[\![\mathsf{snd}(c)]\!]_{\Gamma} = \pi_2 \circ [\![c]\!]_{\Gamma}$$

$$[\![\mathsf{app}(f, a)]\!]_{\Gamma} = eval \circ \langle [\![f]\!]_{\Gamma}, [\![a]\!]_{\Gamma} \rangle$$

$$[\![\lambda x^A.\varphi(x)]\!]_{\Gamma} = curry([\![\varphi(x)]\!]_{\Gamma,x})$$

Element-Free Interpretation of Terms (III)

$$[\![c]\!]_\Gamma = [\![c]\!]^\rightarrow \circ t_{[\![\Gamma]\!]}$$

$$[\![x_i]\!]_\Gamma = \pi_i$$

$$[\![\mathsf{pair}(a,b)]\!]_\Gamma = \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle$$

$$[\![\mathsf{fst}(c)]\!]_\Gamma = \pi_1 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{snd}(c)]\!]_\Gamma = \pi_2 \circ [\![c]\!]_\Gamma$$

$$[\![\mathsf{app}(f,a)]\!]_\Gamma = eval \circ \langle [\![f]\!]_\Gamma, [\![a]\!]_\Gamma \rangle$$

$$[\![\lambda x^A.\varphi(x)]\!]_\Gamma = curry([\![\varphi(x)]\!]_{\Gamma,x})$$

Equations

The properties of the functions we introduced can be used to prove that the type lambda calculus equations hold

Equations

The properties of the functions we introduced can be used to prove that the type lambda calculus equations hold

$\mathsf{fst}(\mathsf{pair}(a,b)) \underset{\Gamma}{=} a$ for all $a \in \mathcal{T}_A$, $b \in \mathcal{T}_B$:

$$\begin{aligned}
[\![\mathsf{fst}(\mathsf{pair}(a,b))]\!]_\Gamma &= \pi_1 \circ [\![\mathsf{pair}(a,b)]\!]_\Gamma \\
&= \pi_1 \circ \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle \\
&= [\![a]\!]_\Gamma
\end{aligned}$$

Equations

The properties of the functions we introduced can be used to prove that the type lambda calculus equations hold

$\mathsf{fst}(\mathsf{pair}(a,b)) \underset{\Gamma}{=} a$ for all $a \in \mathcal{T}_A$, $b \in \mathcal{T}_B$:

$$\begin{aligned} [\![\mathsf{fst}(\mathsf{pair}(a,b))]\!]_\Gamma &= \pi_1 \circ [\![\mathsf{pair}(a,b)]\!]_\Gamma \\ &= \pi_1 \circ \langle [\![a]\!]_\Gamma, [\![b]\!]_\Gamma \rangle \\ &= [\![a]\!]_\Gamma \end{aligned}$$

$\mathsf{app}(\lambda x^A.\varphi(x), a) \underset{\Gamma}{=} \varphi(a)$ for all $a \in \mathcal{T}_A$ (subst. for x):

$$\begin{aligned} [\![\mathsf{app}(\lambda x^A.\varphi(x), a)]\!]_\Gamma &= eval \circ \langle [\![\lambda x^A.\varphi(x)]\!]_\Gamma, [\![a]\!]_\Gamma \rangle \\ &= eval \circ \langle curry([\![\varphi(x)]\!]_{\Gamma,x}), [\![a]\!]_\Gamma \rangle \\ &= eval \circ (curry([\![\varphi(x)]\!]_{\Gamma,x} \times id) \circ \langle id, [\![a]\!]_\Gamma \rangle \\ &= [\![\varphi(x)]\!]_{\Gamma,x} \circ \langle id, [\![a]\!]_\Gamma \rangle \\ &= [\![\varphi(a)]\!]_\Gamma \end{aligned}$$

Cartesian Closed Categories

A category $\mathcal{C}$ is a cartesian closed category (CCC) if

(1) $\mathcal{C}$ has a terminal object
(An object T s.t. for every object X, there is exactly one morphism $t_X : X \to T$)

(2) $\mathcal{C}$ has finite products
(For any objects $X_1, \ldots, X_k$, there is an object $X_1 \times \cdots \times X_k$ and morphisms $\pi_i : X_1 \times \cdots \times X_k \to X_i$ such that ...)

(3) $\mathcal{C}$ has exponentials
(For any objects X and Y, there is an object Y^X and a morphism $eval : Y^X \times X \to Y$ such that ...)

The category of sets (with functions) is a CCC

Further Remarks

- For any cartesian closed category $\mathcal{C}$, can “extract” a typed lambda calculus
  - Types are objects of $\mathcal{C}$
  - Terms are a certain class of morphisms of $\mathcal{C}$
  - Called the *internal language* of $\mathcal{C}$
  - **Theorem:** proving the a diagram commutes in $\mathcal{C}$ is equivalent to proving an equation in the internal language of $\mathcal{C}$
- The relationship between cartesian closed categories and typed lambda calculi is in fact deeper
  - **Theorem:** Cartesian closed categories and typed lambda calculi are equivalent as categories

Beyond Typed Lambda Calculi

- Can play the same game with the untyped lambda calculus
  - Need one-object cartesian closed categories without a terminal object
  - Much harder to come by (see Wojtek's talk)
- Can play the same game with type theory (typed lambda calculus with truth values and logical constants)
  - Need *topoi*
  - Every topos is cartesian closed